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# Geometric properties of certain meromorphic functions

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**Abstract**

In this paper, we aim at investigating several geometric properties of the solutions of the following differential equations:

$$w''(z)+a(z)w'(z)+b(z)w(z)=0,$$

where the functions $a(z)$ and $b(z)$ are meromorphic in the punctured disk $\mathbb{D}=\{z:0<|z|<1\}$.

## 1 Introduction

Let $\Sigma$ be the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n \tag{1.1}$$

which are meromorphic in the punctured disk $\mathbb{D}=\{z:0<|z|<1\}$.

A function $f(z)\in\Sigma$ is said to meromorphic starlike of order $\alpha$ in $\mathbb{D}$ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<-\alpha \qquad (z\in\mathbb{D}) \tag{1.2}$$

for some $\alpha$ $(0\leqq\alpha<1)$. We denoted by $\Sigma\mathcal{S}_0^*(\alpha)$ the subclass of $\Sigma$ consisting of all such functions.

## 2 A class of bounded functions

We begin with the definition and lemma.

**Definition 1** Let $\mathcal{H}_J$ be the class of complex functions $h(s,t)$ satisfying:
$(i)$ $h(s,t)$ is continuous in a domain $\mathbb{D}\subset\mathbb{C}\times\mathbb{C}$,
$(ii)$ $(0,0)\in\mathbb{D}$ and $|h(0,0)|<J$ $(J>0)$,
$(iii)$ $|h(Je^{i\theta},Ke^{i\theta})|\geqq J$ when $(Je^{i\theta},Ke^{i\theta})\in\mathbb{D}$, $\theta$ is real and $K\geqq J$.

**Definition 2** Let $h \in \mathcal{H}_J$ with corresponding domain $\mathbb{D}$. We denote by $\mathcal{B}_J(h)$ the class of functions $u(z) = u_1 z + u_2 z^2 + \cdots$ which are analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and satisfy

(i) $(u(z), zu'(z)) \in \mathbb{D}$,

(ii) $|h(u(z), zu'(z))| < J \quad (z \in \Delta)$.

**Lemma 1** ([3]) *Let $h \in \mathcal{H}_J$ and $b(z)$ be an analytic function in $\Delta$ with $|b(z)| < J$. If the differential equation*

$$h(u(z), zu'(z)) = b(z) \tag{2.1}$$

*has a solution $u(z)$ analytic in $\Delta$, then $|u(z)| < J$.*

**Lemma 2** ([1]) *If $f(z) \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in $\mathbb{D}$ and*

$$-\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} < 4 - \beta \qquad (z \in \Delta), \tag{2.2}$$

*then*

$$-\mathrm{Re}\{z^2 f'(z)\} > \frac{1}{5 - 2\beta} \qquad (z \in \Delta), \tag{2.3}$$

*that is, $f(z)$ is meromorphic close-to-convex of order $\dfrac{1}{5-2\beta}$, where $\dfrac{3}{2} \leqq \beta < 2$.*

# 3 Main results

First, we prove

**Theorem 1** *Let $w(z)$, $a(z) \in \Sigma$ and $b(z)$ are meromorphic in $\mathbb{D}$ with*

$$\left| z^2 \left( b(z) - \frac{1}{2}a'(z) - \frac{1}{4}(a(z))^2 \right) \right| < \frac{1}{2} \qquad (z \in \mathbb{D}) \tag{3.1}$$

*and*

$$\mathrm{Re}\{za(z)\} \geqq 2 + 2\alpha \qquad (0 \leqq \alpha < 1).$$

*Also, let $w(z)$ be the solution of the following second order linear differential equation*

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0. \tag{3.2}$$

*Then $w(z)$ is meromorphic starlike of order $\alpha$.*

*Proof.* Put $w(z) = e^{-\frac{1}{2}\int a(\xi)d\xi} v(z)$. Then (3.2) leads to the normal form

$$v''(z) + \left( b(z) - \frac{1}{2}a'(z) - \frac{1}{4}(a(z))^2 \right) v(z) = 0. \tag{3.3}$$

If we put

$$u(z) = \frac{zv'(z)}{v(z)} - \frac{1}{2} \qquad (z \in \mathbb{D}), \tag{3.4}$$

then $u(z)$ is analytic in $\Delta$ and (3.3) becomes

$$\left(u(z)\right)^2 + zu'(z) - \frac{1}{4} = -z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}\left(a(z)\right)^2\right), \tag{3.5}$$

or equivalently

$$h\left(u(z), zu'(z)\right) = -z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}\left(a(z)\right)^2\right), \tag{3.6}$$

where $h(s,t) = s^2 + t - \frac{1}{4}$. It is easy to check $h(s,t) \in \mathcal{H}_{\frac{1}{2}}$, that is

(*i*) $h(s,t)$ is continuous in $\mathbb{C} \times \mathbb{C}$,
(*ii*) $|h(0,0)| = \frac{1}{4} < \frac{1}{2}$,
(*iii*) $\left|h\left(\frac{1}{2}e^{i\theta}, Ke^{i\theta}\right)\right| \geqq \frac{1}{2} \quad \left(K \geqq \frac{1}{2}\right)$.

From assumption, we have

$$\left|-z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}\left(a(z)\right)^2\right)\right| < \frac{1}{2} \qquad (z \in \mathbb{D}).$$

By using Lemma 1, we have $|u(z)| < \frac{1}{2} \quad (z \in \Delta)$. Therefore, we obtain

$$\left|\frac{zv'(z)}{v(z)} - \frac{1}{2}\right| < \frac{1}{2} \qquad (z \in \Delta).$$

This implies

$$0 < \operatorname{Re}\left\{\frac{zv'(z)}{v(z)}\right\} < 1 \qquad (z \in \Delta).$$

From $w(z) = e^{-\frac{1}{2}\int a(\xi)\,d\xi}\, v(z)$, we have

$$\exp\left(\frac{1}{2}\int a(\xi)\,d\xi\right) w(z) = v(z). \tag{3.7}$$

Logarithmically differentiating of (3.7) leads to

$$\frac{zw'(z)}{w(z)} = \frac{zv'(z)}{v(z)} - \frac{1}{2}za(z). \tag{3.8}$$

Combining (3.8) and $\operatorname{Re}\{za(z)\} \geqq 2 + 2\alpha \quad (0 \leqq \alpha < 1)$, we obtain

$$\operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} = \operatorname{Re}\left\{\frac{zv'(z)}{v(z)}\right\} - \frac{1}{2}\operatorname{Re}\{za(z)\} < 1 - \frac{1}{2}(2 + 2\alpha) = -\alpha \qquad (z \in \mathbb{D}),$$

that is, $w(z)$ is meromorphic starlike of order $\alpha$. $\square$

**Example 1** In Theorem 1, let $a(z) = \dfrac{2}{z}$ and $b(z) = \dfrac{1}{2}$. The solution of

$$w''(z) + \frac{2}{z}w'(z) + \frac{1}{2}w(z) = 0 \tag{3.9}$$

is given by $w(z) = \dfrac{\cos\frac{z}{\sqrt{2}}}{z}$. This solution $w(z)$ is meromorphic starlike function.

Next, we prove

**Theorem 2** *Let $w(z), Q(z) \in \Sigma$. We consider the following second order differential equation.*

$$w''(z) + Q(z)w(z) = 0 \qquad (z \in \mathbb{D}). \tag{3.10}$$

*If*

$$\mathrm{Re}\left\{Q(z)\,\frac{zw(z)}{w'(z)}\right\} < 4 - \beta \qquad (z \in \mathbb{D}),$$

*then we have*

$$-\mathrm{Re}\{z^2 w'(z)\} > \frac{1}{5 - 2\beta} \qquad \left(\frac{3}{2} \leqq \beta < 2\right).$$

*Proof.* From (3.10), we have

$$Q(z)\,\frac{zw(z)}{w'(z)} = -\frac{zw''(z)}{w'(z)}. \tag{3.11}$$

Applying Lemma 2 to (3.11), we can prove Theorem 2. □

**Example 2** In Theorem 2, let $Q(z) = -\dfrac{2}{z^2}$. A solution of

$$w''(z) - \frac{2}{z^2}w(z) = 0$$

is give by $w(z) = \dfrac{1}{z} + \dfrac{3}{50}z^2$. Then

$$\mathrm{Re}\left\{Q(z)\,\frac{zw(z)}{w'(z)}\right\} < 2.404\cdots < \frac{5}{2}$$

and

$$-\mathrm{Re}\{z^2 w'(z)\} > 0.88 > \frac{1}{2}.$$

Therefore, $w(z)$ is meromorphic close-to-convex function.

**Remark 1** Let $\mathcal{MC}(\alpha)$ be the subclass of $\Sigma$ consisting of functions $f(z)$ which satisfy

$$-\mathrm{Re}\{z^2 f'(z)\} > \alpha \qquad (z \in \Delta) \tag{3.12}$$

for some $\alpha$ $(0 \leqq \alpha < 1)$. A function $f(z) \in \mathcal{MC}(\alpha)$ is meromorphic close-to-convex of order $\alpha$ in $\mathbb{D}$.

## References

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